

Baxter Algebras and the Umbral Calculus ^{*}

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Abstract

We apply Baxter algebras to the study of the umbral calculus. We give a characterization of the umbral calculus in terms of Baxter algebra. This characterization leads to a natural generalization of the umbral calculus that include the classical umbral calculus in a family of λ -umbral calculi parameterized by λ in the base ring.

1 Introduction

Baxter algebra and the umbral calculus are two areas in mathematics that have interested Rota throughout his life time and in which he has made prominent contributions. We will show in this paper that these two areas are intimately related.

The umbral calculus is the study and application of sequences of polynomials of binomial type and other related sequences. More precisely, a sequence $\{p_n(x) \mid n \in \mathbb{N}\}$ of polynomials in $C[x]$ is called a sequence of binomial type if

$$p_n(x+y) = \sum_{k=0}^n p_k(x)p_{n-k}(y), \forall y \in C, n \in \mathbb{N}.$$

Such sequences have fascinated mathematicians since the 19th century and include many of the most well-known sequences, such as the ones named after Abel, Bernoulli, Euler, Hermite and Mittag-Leffler. Even though polynomials of binomial type proved to be useful in several areas of mathematics, the foundation of umbral calculus was not firmly established for over a century since its first introduction. This situation changed completely in 1964 when G.-C. Rota[**Ro1**] indicated that the theory can be rigorously formulated in terms of the algebra of functionals defined on the polynomials, later known as the umbral algebra. Rota's pioneer work was completed in the next

^{*}MSC: 16W99, 05A50

decade by Rota and his collaborators [RKO, RR, Rom]. Since then, there have been a number of generalizations of the umbral calculus [Rom, Ch, Lo, Me, Ve].

During the same period of time in which Rota embarked on laying down the foundation of the umbral calculus, he also started the algebraic study of Baxter algebra which was first introduced by Baxter in connection with fluctuation theory in probability [Ba]. Fundamental to the study of Baxter algebra are the important works of Rota [Ro2] and Cartier [Ca] that gave constructions of free Baxter algebras. By using a generalization of shuffle product in topology and geometry, the present author and W. Keigher gave another construction of free Baxter algebras [G-K1, G-K2]. This construction is applied to the further study of free Baxter algebras [Gu1, Gu2, AGKO].

Our first purpose in this paper is to give a characterization of umbral calculus in terms of free Baxter algebras. We show that the umbral algebra is the free Baxter algebra of weight zero on the empty set. We also characterize the polynomial sequences studied in the umbral calculus in terms of operations in free Baxter algebras.

The second purpose of this paper is to use the free Baxter algebra formulation of the umbral calculus we have obtained to give a generalization of the umbral calculus, the λ -umbral calculus, for each constant λ in the base ring C . The umbral calculus of Rota is the special case when $\lambda = 0$.

For simplicity, we only consider sequences of binomial type in this paper. The study of other sequences in the umbral calculus, such as Sheffer sequences and cross sequences, can be similarly generalized to our setting. We hope to explore possible roles played by Baxter algebras in other generalizations of the umbral calculus. We also plan to give a formulation of the umbral calculus in terms of coalgebras with operators by combining the approach in this paper and the coalgebra approach in [RR, NS]. These projects will be carried out in subsequent papers.

The layout of this paper is as follows. In section 2, we review the umbral calculus and give a characterization of umbral calculus in terms of Baxter algebra. In section 3, we define sequences of λ -binomial type and formulate the basic theory of the λ -umbral calculus that generalizes the classic theory of Rota. In section 4, under the assumption that C is a \mathbb{Q} -algebra, we give an explicit construction of sequences of λ -binomial type. We also study the relation between the λ -binomial sequences and the classic binomial sequences.

2 The umbral calculus and umbral algebra

2.1 Background on the umbral calculus

We first recall some background on the umbral calculus. See [RKO, Rom] for more details.

Let \mathbb{N} be the set of non-negative integers. Let $C[x]$ be the C -algebra of polynomials with coefficients in C . The main objects to study in the classical umbral calculus

are special sequences of polynomials called sequences of binomial type and Sheffer sequences.

Definition 2.1 1. A sequence $\{p_n(x) \mid n \in \mathbb{N}\}$ of polynomials in $C[x]$ is called a **sequence of binomial type** if

$$p_n(x+y) = \sum_{k=0}^n p_{n-k}(x)p_k(y), \forall y \in C, n \in \mathbb{N}.$$

2. Given a sequence of binomial type $\{p_n(x)\}$, a sequence $\{s_n(x) \mid n \in \mathbb{N}\}$ of polynomials in $C[x]$ is called a **Sheffer sequence** relative to $\{p_n(x)\}$ if

$$s_n(x+y) = \sum_{k=0}^n p_{n-k}(x)s_k(y), \forall y \in C, n \in \mathbb{N}.$$

In order to describe these sequences, Rota and his collaborators studied the dual of the C -module $C[x]$ and endowed the dual with a C -algebra structure, called the umbral algebra. It can be defined as follows. Let $\{t_n \mid n \in \mathbb{N}\}$ be a sequence of symbols. Let \mathcal{F} be the C -module $\prod_{n \in \mathbb{N}} C t_n$, where the addition and scalar multiplication are defined componentwise. Define a multiplication on \mathcal{F} by assigning

$$t_m t_n = \binom{m+n}{m} t_{m+n}, \quad m, n \in \mathbb{N}. \quad (1)$$

This makes \mathcal{F} into a C -algebra, with t_0 being the identity. The C -algebra \mathcal{F} , together with the basis $\{t_n\}$ is called the **umbral algebra**. When C is a \mathbb{Q} -algebra, it follows from Eq (1) that

$$t_n = \frac{t_1^n}{n!}, \quad n \in \mathbb{N}$$

and so $\mathcal{F} \cong C[[t_1]]$ as a C -algebra. But we want to emphasize the special basis $\{t_n\}$.

One then identifies \mathcal{F} with the dual C -module of $C[x]$ by taking $\{t_n\}$ to be the dual basis of $\{x^n\}$. In other words, t_k is defined by

$$t_k : C[x] \rightarrow C, \quad x^n \mapsto \delta_{k,n}, \quad k, n \in \mathbb{N}.$$

Rota and his collaborators removed the mystery of sequences of binomial type and Sheffer sequences by showing that they have a simple characterization in terms of the umbral algebra.

Let f_n , $n \geq 0$ be a pseudo-basis of \mathcal{F} . This means that $\{f_n, n \geq 0\}$ is linearly independent and generates \mathcal{F} as a topology C -module where the topology on \mathcal{F} is defined by the filtration

$$F^n = \left\{ \sum_{k=1}^{\infty} c_k t_k \mid c_k = 0, k \leq n \right\}.$$

A pseudo-basis f_n , $n \geq 0$ is called a **divided power pseudo-basis** if

$$f_m f_n = \binom{m+n}{m} f_{m+n}, \quad m, n \geq 0.$$

Much of the foundation for the umbral calculus can be summarized in the following theorem.

Theorem 2.2 [RR] *Let C be a \mathbb{Q} -algebra.*

1. *A polynomial sequence $\{p_n(x)\}$ is of binomial type if and only if it is the dual basis of a divided power pseudo-basis of \mathcal{F} .*
2. *Any divided power pseudo-basis of $C[[t]]$ is of the form $f_n(x) = \frac{f^n(t)}{n!}$ for some $f \in C[[t]]$ with $\deg f = 1$ (i.e., $f(t) = \sum_{k=1}^{\infty} c_k t^k$, $c_1 \neq 0$).*

Sheffer sequences can be similarly described.

2.2 Baxter algebras

We will give a characterization of the umbral calculus in terms of Baxter algebras. For this purpose we recall some definitions and basic properties on Baxter algebras. For further details, see [G-K1].

Definition 2.3 *Let A be a C -algebra and let λ be in C .*

1. *A C -linear operator $P : A \rightarrow A$ is called a **Baxter operator of weight λ** if*

$$P(x)P(y) = P(xP(y)) + P(yP(x)) + \lambda P(xy), \quad x, y \in A.$$

2. *The pair (A, P) , where A is a C -algebra and P is a Baxter operator on A of weight λ , is called a **Baxter C -algebra of weight λ** .*

We often suppress λ from the notations when there is no danger of confusion.

Definition 2.4 *Let A be a C -algebra. A free Baxter algebra on A of weight λ is a weight λ Baxter algebra (F_A, P_A) together with a C -algebra morphism $j_A : A \rightarrow F_A$ such that, for any weight λ Baxter algebra (R, P) and any C -algebra morphism $f : A \rightarrow R$, there is a unique weight λ Baxter algebra morphism $\tilde{f} : (F_A, P_A) \rightarrow (R, P)$ with $\tilde{f} \circ j_A = f$.*

Free Baxter algebras were constructed in [G-K1], generalizing the work of Cartier [Ca] and Rota [Ro2]. In the special case when $A = C$, we have

Theorem 2.5 [G-K1] Let $U_\lambda C$ be the direct product $\prod_{n \in \mathbb{N}} Cu_n$ of the rank one free C -modules Cu_n , $n \in \mathbb{N}$.

1. With the product defined by

$$u_m u_n = \sum_{k=0}^m \binom{m+n-k}{n} \binom{n}{k} \lambda^k u_{m+n-k}, \quad m, n \in \mathbb{N},$$

$U_\lambda C$ becomes a C -algebra.

2. The operator

$$P_C : U_\lambda C \rightarrow U_\lambda C, \quad u_n \mapsto u_{n+1}, \quad n \in \mathbb{N}$$

is a Baxter operator of weight λ on the C -algebra $U_\lambda C$. Further, the pair $(U_\lambda C, P_C)$ is the free complete Baxter algebra on C .

Proposition 2.6 Fix $\lambda \in C$. Let $\{v_n\}_n$ be a pseudo-basis of $U_\lambda C$. the following statements are equivalent.

- 1.

$$v_m v_n = \sum_{k=0}^m \binom{m+n-k}{n} \binom{n}{k} \lambda^k v_{m+n-k}, \quad m, n \in \mathbb{N}.$$

2. The operator

$$P_v : U_\lambda C \rightarrow U_\lambda C, \quad v_n \mapsto v_{n+1}, \quad n \in \mathbb{N}$$

is a Baxter operator of weight λ on the C -algebra $U_\lambda C$. Further, the pair $(U_\lambda C, P_v)$ is the free complete Baxter algebra on C .

Proof: The proposition follows immediately from the definition. ■

Definition 2.7 Let $\{v_n\}_n$ be a pseudo-basis of $U_\lambda C$. If any of the equivalent conditions in Proposition 2.6 is true, we call $\{v_n\}_n$ a **λ -divided power pseudo-basis** of $U_\lambda C$.

We can now give a characterization of the umbral algebra and polynomial sequences of binomial type.

Theorem 2.8 1. The umbral algebra is the free Baxter algebra of weight zero on C .

2. Let $\{p_n(x)\}_{n \in \mathbb{N}}$ be a sequence of polynomials in $C[x]$. Then $\{p_n(x)\}_n$ is of binomial type if and only if it is the dual basis of a divided pseudo-basis of $U_\lambda C$.

3 The λ -umbral calculus

We now develop the theory of the λ -umbral calculus. We construct the λ -umbral algebra and establish the relation between the λ -umbral calculus and λ -umbral algebra. In the special case when $\lambda = 0$, we have the theory started by Rota on the umbral calculus.

3.1 Definitions

In view of Theorem 2.8, we will study the following generalization of the umbral calculus. In order to get interesting examples, we will work with $C[[x]]$ instead of $C[x]$.

Definition 3.1 *A sequence $\{p_n(x) \mid n \in \mathbb{N}\}$ of power series in $C[[x]]$ is a sequence of λ -binomial type if*

$$p_n(x+y) = \sum_{k=0}^n \lambda^k \sum_{i=0}^n \binom{n}{i} \binom{i}{k} p_{n+k-i}(x) p_i(y), \forall y \in C, n \in \mathbb{N}.$$

When $\lambda = 0$, we recover the sequences of binomial type. We also note the following symmetric property.

Lemma 3.2 *A sequence $\{p_n(x) \mid n \in \mathbb{N}\}$ of power series in $C[[x]]$ is a sequence of λ -binomial type if and only if*

$$p_n(x+y) = \sum_{k=0}^n \lambda^k \sum_{i=0}^n \binom{n}{i} \binom{i}{k} p_i(x) p_{n+k-i}(y), \forall y \in C, n \in \mathbb{N}.$$

Proof: We have

$$\begin{aligned} \sum_{k=0}^n \lambda^k \sum_{i=0}^n \binom{n}{i} \binom{i}{k} p_{n+k-i}(x) p_i(y) &= \sum_{i=0}^n \sum_{k=0}^n \lambda^k \binom{n}{i} \binom{i}{k} p_{n+k-i}(x) p_i(y) \\ &= \sum_{i=0}^n \sum_{k=0}^i \lambda^k \binom{n}{i} \binom{i}{k} p_{n+k-i}(x) p_i(y) \\ &= \sum_{i=0}^n \sum_{j=n-i}^n \lambda^{j-n+i} \binom{n}{i} \binom{i}{j-n+i} p_j(x) p_i(y) \\ &\quad \text{(use substitution } j = n + k - i, \ k = j - n + i) \\ &= \sum_{i=0}^n \sum_{j=0}^n \lambda^{j-n+i} \binom{n}{i} \binom{i}{j-n+i} p_j(x) p_i(y) \\ &= \sum_{j=0}^n \sum_{i=0}^n \lambda^{j-n+i} \binom{n}{i} \binom{i}{j-n+i} p_j(x) p_i(y) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^n \sum_{\ell=j-n}^j \lambda^\ell \binom{n}{i} \binom{i}{\ell} p_j(x) p_{n+\ell-j}(y) \\
&\quad \text{(using substitution } \ell = j - n + i, i = n + \ell - j) \\
&= \sum_{j=0}^n \sum_{\ell=0}^n \lambda^\ell \binom{n}{i} \binom{i}{\ell} p_j(x) p_{n+\ell-j}(y) \\
&= \sum_{\ell=0}^n \sum_{j=0}^n \lambda^\ell \binom{n}{i} \binom{i}{\ell} p_j(x) p_{n+\ell-j}(y).
\end{aligned}$$

This proves the lemma. ■

Definition 3.3 Fix a $\lambda \in C$. The algebra $U_\lambda C = \prod_n C u_n$, together with the λ -divided power pseudo-basis $\{u_n\}_n$, is called the λ -umbral algebra.

Fix a pseudo-basis $\mathbf{q} = \{q_n(x)\}$ of $C[[x]]$ of λ -binomial type (see §4 for the existence of such sequences). Let $C\langle \mathbf{q} \rangle$ be the submodule of $C[[x]]$ generated by the pseudo-basis \mathbf{q} . As in the case of $\lambda = 0$, we identify $U_\lambda C$ with the dual C -module of $C\langle \mathbf{q} \rangle$ by taking $\{u_n\}$ to be the dual basis of $\{q_n(x)\}$. Thus each element of $U_\lambda C$ can be regarded as a functional on $C\langle \mathbf{q} \rangle$. We denote $\langle \mid \rangle_\lambda$ for the resulting pairing

$$U_\lambda C \otimes C\langle \mathbf{q} \rangle \rightarrow C.$$

It is characterized by

$$\langle u_n \mid q_k \rangle_\lambda = \delta_{k,n} \quad \forall n, k \in \mathbb{N}. \quad (2)$$

Lemma 3.4 The pairing $\langle \mid \rangle_\lambda$ is pseudo-perfect in the sense that

1. for a fixed $u \in U_\lambda C$, if $\langle u \mid f(x) \rangle_\lambda = 0, \forall f(x) \in C\langle \mathbf{q} \rangle$, then $u = 0$, and
2. for a fixed $f(x) \in C\langle \mathbf{q} \rangle$, if $\langle u \mid f(x) \rangle_\lambda = 0, \forall u \in U_\lambda C$, then $f(x) = 0$.

Proof: It follows from Eq.(2) and the fact that $\{u_n\}$ and $\{q_k\}$ are pseudo-bases. ■

3.2 Basic properties

The following elementary properties can be proved in the same way as in the classical case.

Proposition 3.5 Let $\{p_n(x)\} \subset C\langle \mathbf{q} \rangle$ be a sequence of λ -binomial type. Let $\{v_n\} \subset U_\lambda C$ be the dual basis of $\{p_n(x)\}$.

1. **(The Expansion Theorem)** For any $u \in U_\lambda C$,

$$u = \sum_{n=0}^{\infty} \langle u \mid p_n(x) \rangle v_n.$$

2. **(The Polynomial Expansion Theorem)** For any $p(x) \in C\langle \mathfrak{q} \rangle$, we have

$$p(x) = \sum_{n=0}^{\infty} \langle v_n \mid p(x) \rangle p_n(x).$$

Note that $U_\lambda C$ acts on itself on the right, making $U_\lambda C$ a right $U_\lambda C$ -module. This $U_\lambda C$ -module structure, through the pairing $\langle \mid \rangle_\lambda$, makes $C\langle \mathfrak{q} \rangle$ into a left $U_\lambda C$ -module. More precisely, for $u \in U_\lambda C$ and $f \in C\langle \mathfrak{q} \rangle$, the element $uf \in C\langle \mathfrak{q} \rangle$ is characterized by

$$\langle v \mid uf \rangle_\lambda = \langle vu \mid f \rangle_\lambda, \quad \forall v \in U_\lambda C. \quad (3)$$

Proposition 3.6 With the notation above, we have

$$u_k q_n(x) = \sum_{i=0}^k \lambda^i \binom{n}{k} \binom{k}{i} q_{n-k+i}(x), \quad \forall k, n \in \mathbb{N}.$$

Proof: For each $m \in \mathbb{N}$, we have

$$\begin{aligned} \langle u_m \mid u_k s_n \rangle_\lambda &= \langle u_m u_k \mid s_n \rangle_\lambda \\ &= \langle \sum_{i=0}^m \lambda^i \binom{m+k-i}{m} \binom{m}{i} u_{m+k-i} \mid s_n \rangle_\lambda \\ &= \sum_{i=0}^m \lambda^i \binom{m+k-i}{m} \binom{m}{i} \delta_{m+k-i, n} \\ &= \sum_{i=0}^m \lambda^i \binom{m+k-i}{m} \binom{m}{i} \delta_{m, n-k+i} \\ &= \sum_{i=0}^{\infty} \lambda^i \binom{m+k-i}{m} \binom{m}{i} \delta_{m, n-k+i} \quad (i > m \Rightarrow \binom{m}{i} = 0) \\ &= \sum_{i=0}^{\infty} \lambda^i \binom{n}{n-k+i} \binom{n-k+i}{i} \delta_{m, n-k+i} \\ &\quad (\text{the definition of } \delta_{m, n-k+i} \Rightarrow m = n - k + i) \\ &= \sum_{i=0}^{\infty} \lambda^i \binom{n}{n-k+i} \binom{n-k+i}{i} \langle u_m \mid s_{n-k+i} \rangle_\lambda \quad (\text{Eq. (2)}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty} \lambda^i \binom{n}{k} \binom{k}{i} \langle u_m \mid s_{n-k+i} \rangle_{\lambda} \quad \left(\binom{n}{n-k+i} \binom{n-k+i}{i} = \binom{n}{k} \binom{k}{i} \right) \\
&= \sum_{i=0}^k \lambda^i \binom{n}{k} \binom{k}{i} \langle u_m \mid s_{n-k+i} \rangle_{\lambda} \quad (i > k \Rightarrow \binom{k}{i} = 0) \\
&= \langle u_m \mid \sum_{i=0}^k \lambda^i \binom{n}{k} \binom{k}{i} s_{n-k+i} \rangle_{\lambda}.
\end{aligned}$$

Now the proposition follows from Lemma 3.4. ■

Proposition 3.7 *Let $\mathfrak{q} = \{q_n(x)\}$ be the fixed sequence of λ -binomial type. For any $u, v \in U_{\lambda}C$, we have*

$$\langle uv \mid q_n(x) \rangle_{\lambda} = \sum_{k=0}^n \sum_{i=0}^n \lambda^k \binom{n}{i} \binom{i}{k} \langle u \mid q_{n+k-i}(x) \rangle_{\lambda} \langle v \mid q_i(x) \rangle_{\lambda}. \quad (4)$$

Proof: By Proposition 3.5, we have

$$u = \sum_{n=0}^{\infty} \langle u \mid q_n(x) \rangle_{\lambda} u_n,$$

and

$$v = \sum_{n=0}^{\infty} \langle v \mid q_n(x) \rangle_{\lambda} u_n.$$

Then we have

$$\begin{aligned}
uv &= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \langle u \mid q_k(x) \rangle_{\lambda} \langle v \mid q_{m-k}(x) \rangle_{\lambda} u_k u_{m-k} \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \langle u \mid q_k(x) \rangle_{\lambda} \langle v \mid q_{m-k}(x) \rangle_{\lambda} \sum_{i=0}^k \lambda^i \binom{m-i}{k} \binom{k}{i} u_{m-i} \\
&\quad \text{(Proposition 2.6)} \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \langle u \mid q_k(x) \rangle_{\lambda} \langle v \mid q_{m-k}(x) \rangle_{\lambda} \sum_{j=m-k}^m \lambda^{m-j} \binom{j}{k} \binom{k}{m-j} u_j \\
&\quad \text{(replacing } i \text{ by } m-j) \\
&= \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=m-j}^m \langle u \mid q_k(x) \rangle_{\lambda} \langle v \mid q_{m-k}(x) \rangle_{\lambda} \lambda^{m-j} \binom{j}{k} \binom{k}{m-j} u_j \\
&\quad \text{(exchanging the second and the third sum)}
\end{aligned}$$

$$= \sum_{j=0}^{\infty} \left(\sum_{m=j}^{\infty} \sum_{k=m-j}^m < u \mid q_k(x) >_{\lambda} < v \mid q_{m-k}(x) >_{\lambda} \lambda^{m-j} \binom{j}{k} \binom{k}{m-j} \right) u_j$$

(exchanging the first and the second sum).

Since $\binom{k}{m-j} = 0$ for $m > k + j$ and $\binom{j}{k} = 0$ for $k > j$, we have

$$\begin{aligned} & \sum_{m=j}^{\infty} \sum_{k=m-j}^m < u \mid q_k(x) >_{\lambda} < v \mid q_{m-k}(x) >_{\lambda} \lambda^{m-j} \binom{j}{k} \binom{k}{m-j} \\ &= \sum_{m=j}^{2j} \sum_{k=m-j}^m \lambda^{m-j} \binom{j}{k} \binom{k}{m-j} < u \mid q_k(x) >_{\lambda} < v \mid q_{m-k}(x) >_{\lambda} \\ &= \sum_{t=0}^j \sum_{k=t}^{t+j} \lambda^t \binom{j}{k} \binom{k}{t} < u \mid q_k(x) >_{\lambda} < v \mid q_{t+j-k}(x) >_{\lambda} \\ & \quad \text{(replacing } m \text{ by } t + j) \\ &= \sum_{t=0}^j \sum_{k=0}^j \lambda^t \binom{j}{k} \binom{k}{t} < u \mid q_k(x) >_{\lambda} < v \mid q_{t+j-k}(x) >_{\lambda} \\ & \quad (k < t \Rightarrow \binom{k}{t} = 0 \text{ and } k > j \Rightarrow \binom{j}{k} = 0). \end{aligned}$$

Thus

$$uv = \sum_{j=0}^{\infty} \left(\sum_{t=0}^j \sum_{k=0}^j \lambda^t \binom{j}{k} \binom{k}{t} < u \mid q_k(x) >_{\lambda} < v \mid q_{t+j-k}(x) >_{\lambda} \right) u_j.$$

Applying this to $q_n(x)$ and using Eq. (2), we have

$$< uv \mid q_n(x) >_{\lambda} = \sum_{t=0}^n \sum_{k=0}^n \lambda^t \binom{n}{k} \binom{k}{t} < u \mid q_k(x) >_{\lambda} < v \mid q_{n+t-k}(x) >_{\lambda} .$$

Exchanging u and v in the equation, we have

$$\begin{aligned} & < uv \mid q_n(x) >_{\lambda} = < vu \mid q_n(x) >_{\lambda} \\ &= \sum_{t=0}^n \sum_{k=0}^n \lambda^t \binom{n}{k} \binom{k}{t} < v \mid q_k(x) >_{\lambda} < u \mid q_{n+t-k}(x) >_{\lambda} \\ &= \sum_{t=0}^n \sum_{k=0}^n \lambda^t \binom{n}{k} \binom{k}{t} < u \mid q_{n+t-k}(x) >_{\lambda} < v \mid q_k(x) >_{\lambda} . \end{aligned}$$

This proves the proposition. ■

Proposition 3.8 *Let $\{p_n(x)\} \subset C\langle \mathbf{q} \rangle$ be a sequence of λ -binomial type and let u and v be in $U_\lambda C$. Then*

$$\langle uv \mid p_n(x) \rangle_\lambda = \sum_{i=0}^n \sum_{j=0}^n \lambda^i \binom{n}{j} \binom{j}{i} \langle u \mid p_{n+i-j}(x) \rangle_\lambda \langle v \mid p_j(x) \rangle_\lambda .$$

Proof: We follow the case when $\lambda = 0$ [RR]. Let $C[[x, y]]$ be the C -module of power series in the variables x and y . Since $C[[x, y]] \cong C[[x]] \otimes_C C[[y]]$ and $q_n(x)$ is a pseudo-basis of $C[[x]]$, elements of the form $q_i(x)q_j(y)$, $i, j \in \mathbb{N}$ form a pseudo-basis of $C[[x, y]]$. Thus any element $p(x, y)$ of $C[[x, y]]$ can be expressed uniquely in the form

$$p(x, y) = \sum_{i,j} c_{i,j} q_i(x) q_j(y), \quad c_{i,j} \in C.$$

For $u \in U_\lambda C$, define

$$u_x p(x, y) = \sum_{i,j} c_{i,j} \langle u \mid q_i(x) \rangle_\lambda q_j(y)$$

and

$$u_y p(x, y) = \sum_{i,j} c_{i,j} q_i(x) \langle u \mid q_j(y) \rangle_\lambda .$$

Then by Proposition 3.7 and the fact that $q_n(x)$ is a λ -binomial sequence, we have

$$\begin{aligned} \langle uv \mid q_n(x) \rangle_\lambda &= \sum_{i=0}^n \sum_{j=0}^n \lambda^i \binom{n}{j} \binom{j}{i} \langle u \mid q_{n+i-j}(x) \rangle_\lambda \langle v \mid q_j(x) \rangle_\lambda \\ &= u_x v_y q_n(x + y). \end{aligned}$$

Since $q_n(x)$ is a basis of $C\langle \mathbf{q} \rangle$, by the C -linearity of the maps $\langle uv \mid \cdot \rangle_\lambda$, u_x and v_y , we have

$$\langle uv \mid p(x) \rangle_\lambda = u_x v_y p(x + y)$$

for any $p(x) \in C\langle \mathbf{q} \rangle$. Since $p_n(x)$ is of λ -binomial type, we obtain

$$\begin{aligned} \langle uv \mid p_n(x) \rangle_\lambda &= u_x v_y p_n(x + y) \\ &= u_x v_y \left(\sum_{i=0}^n \sum_{j=0}^n \lambda^i \binom{n}{j} \binom{j}{i} p_{n+i-j}(x) p_j(y) \right) \\ &= \sum_{i=0}^n \sum_{j=0}^n \lambda^i \binom{n}{j} \binom{j}{i} \langle u \mid p_{n+i-j}(x) \rangle_\lambda \langle v \mid p_j(x) \rangle_\lambda . \end{aligned}$$

■

Let $c \in C$. Define the **shift operator** E^c on $C\langle \mathbf{q} \rangle$ by

$$(E^c f)(x) = f(x + c).$$

An operator L on $C\langle \mathbf{q} \rangle$ is called **shift invariant** if

$$E^c L = L E^c, \quad \forall c \in C.$$

Proposition 3.9 *The elements in $U_\lambda C$, regarded as operators on $C\langle \mathbf{q} \rangle$, are shift invariant.*

Proof: We only need to check that u_k are shift invariant when applied to $q_n(x)$. Then since $q_n(x)$ is a basis of $C\langle \mathbf{q} \rangle$, u_k is shift invariant on $C\langle \mathbf{q} \rangle$. Since u_k is a pseudo-basis of $U_\lambda C$, we see further that every element in $U_\lambda C$ is shift invariant on $C\langle \mathbf{q} \rangle$, as is needed.

Fix $k, n \geq 0$. We have

$$\begin{aligned} E^c u_k q_n(x) &= \sum_{s=0}^k \lambda^s \binom{n}{k} \binom{k}{s} q_{n-k+s}(n+c) \\ &= \sum_{s=0}^k \lambda^s \binom{n}{k} \binom{k}{s} \sum_{\ell=0}^{n-k+s} \binom{n-k+s}{i} \binom{i}{\ell} q_{n-k+s+\ell-i} q_i(c) \\ &= \sum_{i=0}^n \sum_{s=0}^k \sum_{\ell=0}^n \lambda^{s+\ell} \binom{n}{k} \binom{k}{s} \binom{n-k+s}{i} \binom{i}{\ell} q_{n+\ell+s-k-i} q_i(c) \\ &= \sum_{i=0}^n \sum_{w=0}^{n+k} \lambda^w \binom{n}{k} \sum_{s=0}^w \binom{k}{s} \binom{n-k+s}{i} \binom{i}{w-s} q_{n+w-k-i}(x) q_i(c) \\ &\quad (\text{letting } w = \ell + s). \end{aligned}$$

Also

$$\begin{aligned} u_k E^c q_n(x) &= u_k q_n(x+c) \\ &= u_k \sum_{\ell=0}^n \lambda^\ell \sum_{i=0}^n \binom{n}{i} \binom{i}{\ell} q_{n+\ell-i}(x) q_i(c) \\ &= \sum_{\ell=0}^n \lambda^\ell \sum_{i=0}^n \binom{n}{i} \binom{i}{\ell} \sum_{s=0}^k \lambda^s \binom{n+\ell-i}{k} \binom{k}{s} q_{n+\ell-i-k+s} q_i(c) \\ &= \sum_{i=0}^n \sum_{s=0}^k \sum_{\ell=0}^n \lambda^{\ell+s} \binom{n}{i} \binom{i}{\ell} \binom{n+\ell-i}{k} \binom{k}{s} q_{n+\ell-i-k+s} q_i(c) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^n \sum_{w=0}^{n+k} \lambda^w \binom{n}{k} \sum_{s=0}^w \binom{n+w-s-i}{w-s} \binom{n-k}{i-w+s} \binom{k}{s} q_{n+w-i-k} q_i(c) \\
&\quad (\text{letting } w = \ell + s) \\
&= \sum_{i=0}^n \sum_{w=0}^{n+k} \lambda^w \binom{n}{k} \sum_{s=0}^w \binom{k}{w-s} \binom{n+s-i}{s} \binom{n-k}{i-s} q_{n+w-i-k} q_i(c) \\
&\quad (\text{replacing } s \text{ by } w-s).
\end{aligned}$$

Thus we only need to prove

$$\sum_{s=0}^w \binom{k}{s} \binom{n-k+s}{i} \binom{i}{w-s} = \sum_{s=0}^w \binom{k}{w-s} \binom{n+s-i}{s} \binom{n-k}{i-s} \quad (5)$$

for $n, k, i, w \geq 0$. With the help of the Zeilberger algorithm [PWZ], we can verify that both sides of the equation satisfy the same recursive relation

$$\begin{aligned}
&((k+i-w) * (k-n+i-w-1) \\
&+ (k^2 - k * n + k * i - 3 * k * w - n * i + 2 * n * w + i^2 - 3 * i * w \\
&+ 2 * w^2 - 4 * k + 2 * n - 4 * i + 5 * w + 3) * W \\
&- (w+2) * (k-n+i-w-2) * W^2) F(n, k, i, w) = 0
\end{aligned}$$

for $n, k, i \geq 0$. Here W is the shift operator $WF(n, k, i, w) = F(n, k, i, w+1)$. we can also easily verify Eq.(5) directly for $w = 0, 1$. Therefore Eq. (5) is verified. This completes the proof of Proposition 3.9. ■

3.3 Sequences of λ -binomial type and Baxter bases

Now we are ready to state our main theorem in the theory of the λ -umbral calculus.

Theorem 3.10 *Let $\{p_n(x)\}_{n \in \mathbb{N}}$ be a basis of $C\langle \mathbf{q} \rangle$. The following statements are equivalent.*

1. *The sequence $\{p_n(x)\}$ is a sequence of λ -binomial type.*
2. *The sequence $\{p_n(x)\}$ is the dual basis of a λ -divided power pseudo-basis of $U_\lambda C$.*
3. *For all u and v in $U_\lambda C$,*

$$\langle uv \mid p_n(x) \rangle_\lambda = \sum_{i=0}^n \sum_{j=0}^n \lambda^i \binom{n}{j} \binom{j}{i} \langle u \mid p_{n+i-j}(x) \rangle_\lambda \langle v \mid p_j(x) \rangle_\lambda.$$

Remark 3.1 *As in the case when $\lambda = 0$, the third statement in the theorem has the following interpretation in terms of coalgebra.*

3' The C -linear map $q_n(x) \mapsto p_n(x)$, $n \in \mathbb{N}$, defines an automorphism of the C -coalgebra $C\langle \mathbf{q} \rangle$. Here the coproduct

$$\Delta : C\langle \mathbf{q} \rangle \rightarrow C\langle \mathbf{q} \rangle \otimes C\langle \mathbf{q} \rangle$$

is defined by first assigning

$$\Delta(q_n(x)) = \sum_{i=0}^n \sum_{j=0}^n \lambda^i \binom{n}{j} \binom{j}{i} q_{n+i-j}(x) \otimes q_j(x), \quad n \in \mathbb{N}$$

and then extend Δ to $C\langle \mathbf{q} \rangle$ by C -linearity.

Proof: $(1 \Rightarrow 3)$ has been proved in Proposition 3.8.

$(3 \Rightarrow 2)$: Let v_n be the dual basis of $p_n(x)$ in $U_\lambda C$. Then we have

$$\begin{aligned} \langle v_m v_n \mid p_k(x) \rangle_\lambda &= \sum_{i=0}^k \sum_{j=0}^k \lambda^i \binom{k}{j} \binom{j}{i} \langle v_m \mid p_{k+i-j}(x) \rangle_\lambda \langle v_n \mid p_j(x) \rangle_\lambda \\ &= \sum_{i=0}^k \sum_{j=0}^k \lambda^i \binom{k}{j} \binom{j}{i} \delta_{m,k+i-j} \delta_{n,j} \\ &= \sum_{i=0}^k \lambda^i \binom{k}{m} \binom{m}{i} \delta_{m,k+i-n} \quad (\delta_{n,j} = 0 \text{ for } j \neq m) \\ &= \sum_{i=0}^k \lambda^i \binom{k}{m} \binom{m}{i} \delta_{m+n-k,i} \\ &= \lambda^{m+n-k} \binom{k}{m} \binom{m}{m+n-k}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\langle \sum_{i=0}^m \lambda^i \binom{m+n-i}{m} \binom{m}{i} v_{m+n-i} \mid p_k(x) \rangle_\lambda \\ &= \sum_{i=0}^m \lambda^i \binom{m+n-i}{m} \binom{m}{i} \delta_{m+n-i,k} \\ &= \sum_{i=0}^m \lambda^i \binom{m+n-i}{m} \binom{m}{i} \delta_{m+n-k,i} \\ &= \lambda^{m+n-k} \binom{k}{m} \binom{m}{m+n-k}. \end{aligned}$$

So by Lemma 3.4,

$$v_m v_n = \sum_{i=0}^m \lambda^i \binom{m+n-i}{m} \binom{m}{i} v_{m+n-i}$$

and v_n is a λ -divided power pseudo-basis of $U_\lambda C$.

(2 \Rightarrow 1) Let $p_n(x)$ be the dual basis of a λ -divided pseudo-basis v_n of $U_\lambda C$. Let y be a variable and consider the C -algebra $C_1 \stackrel{\text{def}}{=} C[[y]]$. Regarding $p_n(x)$ as elements in $C_1[[x]]$, the sequence $p_n(x)$ is still of λ -binomial type. Also, let

$$U_\lambda C_1 = \prod_{n \in \mathbb{N}} C_1 u_n$$

be the C_1 -algebra obtained from $U_\lambda C$ by scalar extension. Then v_n is also a λ -divided power pseudo-basis of $U_\lambda C_1$. Further, the C -linear perfect pairing (2) extends to a C_1 -linear perfect pairing between $C_1 \langle \mathfrak{q} \rangle$ and $U_\lambda C_1$. Under this pairing, $\{p_n(x)\}$ is still the dual basis of $\{v_n\}$. Note that all previous results applies when C is replaced by C_1 since we did not put any restrictions on C . With this in mind, we have

$$\begin{aligned} p_n(x+y) &= \sum_{k=0}^{\infty} \langle u_k \mid p_n(x+y) \rangle_{\lambda} p_k(x) \quad (\text{Lemma 3.5}) \\ &= \sum_{k=0}^{\infty} \langle u_k \mid E^y p_n(x) \rangle_{\lambda} p_k(x) \quad (\text{definition of } E^y) \\ &= \sum_{k=0}^{\infty} \langle u_k E^y \mid p_n(x) \rangle_{\lambda} p_k(x) \quad (\text{Eq.(3)}) \\ &= \sum_{k=0}^{\infty} \langle E^y u_k \mid p_n(x) \rangle_{\lambda} p_k(x) \quad (\text{Proposition 3.9}) \\ &= \sum_{k=0}^{\infty} \langle E^y \mid u_k p_n(x) \rangle_{\lambda} p_k(x) \quad (\text{Eq. 3}) \\ &= \sum_{k=0}^{\infty} \langle E^y \mid \sum_{i=0}^k \lambda^i \binom{n}{k} \binom{k}{i} p_{n-k+i}(x) \rangle_{\lambda} p_k(x) \quad (\text{Proposition 3.6}) \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^k \lambda^i \binom{n}{k} \binom{k}{i} p_{n-k+i}(y) p_k(x) \quad (\text{definition of } E^y) \\ &= \sum_{k=0}^n \sum_{i=0}^n \lambda^i \binom{n}{k} \binom{k}{i} p_{n-k+i}(y) p_k(x) \quad (k > n \Rightarrow \binom{n}{k} = 0, i > k \Rightarrow \binom{k}{i} = 0) \\ &= \sum_{i=0}^n \sum_{k=0}^n \lambda^i \binom{n}{k} \binom{k}{i} p_k(x) p_{n+i-k}(y) \quad (\text{exchanging the sums}). \end{aligned}$$

■

4 λ -binomial sequences in $C[[x]]$

In this section we study λ -binomial sequences in $C\langle \mathbf{q} \rangle$ when C is a \mathbb{Q} -algebra C . We show that $U_\lambda C$ is isomorphic to $C[[t]]$ regardless of λ . We further show that the pairing in the λ -umbral calculus and the pairing in the classical umbral calculus (when $\lambda = 0$) are compatible. The situation is quite different when C is not a \mathbb{Q} -algebra and will be studied in a subsequent paper.

4.1 The λ -umbral algebra and λ -divided powers

We first describe the λ -umbral algebra $U_\lambda C$.

Definition 4.1 *An power series $f(t) \in C[[t]]$ is call **delta** if $\deg f = 1$, that is, $f(t) = \sum_{k=1}^{\infty} a_k t^k$ with $a_1 \neq 0$.*

Proposition 4.2 *1. Let $f(t)$ be a delta series. Then*

$$\tau_n(f(t)) \stackrel{\text{def}}{=} \frac{f(t)(f(t) - \lambda) \cdots (f(t) - \lambda(n-1))}{n!}, n \geq 0$$

form a λ -divided power pseudo-basis for $C[[t]]$.

2. The map

$$C[[t]] \rightarrow U_\lambda C, \tau_n(t) = \frac{t(t - \lambda) \cdots (t - \lambda(n-1))}{n!} \mapsto t_n, n \geq 0$$

identifies $C[[t]]$ with the λ -umbral algebra $U_\lambda C$.

Proof: 1. Since $f(t)$ is delta, we have $\deg \tau_n(f) = n$. So $\tau_n(f(t))$ form a pseudo-basis for $C[[t]]$. Thus we only need to show that the pseudo-basis is a λ -divided power series. For this we prove by induction on $m \in \mathbb{N}$ that, for any $n \in \mathbb{N}$,

$$\tau_m(f(t))\tau_n(f(t)) = \sum_{k=0}^m \binom{m+n-k}{m} \binom{m}{k} \lambda^k \tau_{m+n-k}(f(t)). \quad (6)$$

The equation clearly holds for $m = 0$. Assume that equation (6) holds for an $m \in \mathbb{N}$ and all $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, we have

$$\begin{aligned} & \sum_{k=0}^{m+1} \binom{m+n+1-k}{m+1} \binom{m+1}{k} \lambda^k \tau_{m+n+1-k}(f(t)) \\ &= \sum_{k=0}^{m+1} \binom{m+n+1-k}{m+1} \binom{m}{k} \lambda^k \tau_{m+n+1-k}(f(t)) \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=0}^{m+1} \binom{m+n+1-k}{m+1} \binom{m}{k-1} \lambda^k \tau_{m+n+1-k}(f(t)) \\
& = \sum_{k=0}^m \binom{m+n+1-k}{m+1} \binom{m+1}{k} \lambda^k \tau_{m+n+1-k}(f(t)) \\
& \quad + \sum_{k=0}^m \binom{m+n-k}{m+1} \binom{m+1}{k} \lambda^{k+1} \tau_{m+n-k}(f(t)) \\
& = \sum_{k=0}^m \binom{m+n-k}{m} \binom{m}{k} \lambda^k \tau_{m+n-k}(f(t)) \left(\frac{f(t) - (m+n-k)\lambda}{m+1} \right) \\
& \quad + \sum_{k=0}^m \binom{m+n-k}{m} \binom{m}{k} \lambda^k \tau_{m+n-k}(f(t)) \left(\frac{(n-k)\lambda}{m+1} \right) \\
& = \sum_{k=0}^m \binom{m+n-k}{m} \binom{m}{k} \lambda^k \tau_{m+n-k}(f(t)) \left(\frac{f(t) - (m+n-k)\lambda}{m+1} - \frac{(n-k)\lambda}{m+1} \right) \\
& = \tau_m(f(t)) \tau_n(f(t)) \left(\frac{f(t) - m\lambda}{m+1} \right) \\
& = \tau_{m+1}(f(t)) \tau_n(f(t)).
\end{aligned}$$

This completes the induction.

2. The statement is clear since $\{\tau_{\lambda,n}(t)\}_n$ is a weight λ λ -divided power pseudo-basis. ■

We next construct a sequence of λ -binomial type. For a given $\lambda \in C$, we use $e_\lambda(x) = \frac{e^{\lambda x} - 1}{\lambda}$ to denote the series $\sum_{k=1}^{\infty} \frac{\lambda^{k-1} x^k}{k!}$. When $\lambda = 0$, we have $e_\lambda(x) = x$.

Proposition 4.3 *Let C be a \mathbb{Q} -algebra. $\{e_\lambda^n(x)\}_n$ is a λ -binomial pseudo-basis for $C[[x]]$.*

Proof: We prove by induction the equation

$$e_\lambda^n(x+c) = \sum_{k=0}^n \lambda^k \sum_{i=0}^n \binom{n}{i} \binom{i}{k} e_\lambda^{n+k-i}(x) e_\lambda^i(c), \forall c \in C, n \in \mathbb{N}. \quad (7)$$

Clearly Eq. (7) holds for $n = 0$. It is also easy to verify the equation

$$e_\lambda(x+c) = e_\lambda(x) + e_\lambda(c) + \lambda e_\lambda(x) e_\lambda(c)$$

which is Eq. (7) when $n = 1$. Using this and the induction hypothesis, we get

$$e_\lambda^{n+1}(x+c) = e(x+c) \sum_{k=0}^n \lambda^k \sum_{i=0}^n \binom{n}{i} \binom{i}{k} e_\lambda^{n+k-i}(x) e_\lambda^i(c)$$

$$\begin{aligned}
&= \sum_{k=0}^n \lambda^k \sum_{i=0}^n \binom{n}{i} \binom{i}{k} \left[e_{\lambda}^{n+1+k-i}(x) e_{\lambda}^i(c) + e_{\lambda}^{n+k-i}(x) e_{\lambda}^{i+1}(c) + \lambda e_{\lambda}^{n+1+k-i} e_{\lambda}^{i+1}(c) \right] \\
&= \sum_{k=0}^n \lambda^k \sum_{i=0}^n \binom{n}{i} \binom{i}{k} e_{\lambda}^{n+1+k-i}(x) e_{\lambda}^i(c) + \sum_{k=0}^n \lambda^k \sum_{i=1}^{n+1} \binom{n}{i-1} \binom{i-1}{k} e_{\lambda}^{n+1+k-i}(x) e_{\lambda}^i(c) \\
&\quad + \sum_{k=1}^{n+1} \lambda^k \sum_{i=1}^{k+1} \binom{n}{i-1} \binom{i-1}{k} e_{\lambda}^{n+2+k-i} e_{\lambda}^i(c) \\
&= \sum_{i=0}^{n+1} \left(\binom{n}{i} + \binom{n}{i-1} \right) e_{\lambda}^{n+1-i}(x) e_{\lambda}^i(c) \\
&\quad + \sum_{k=1}^n \lambda^k \left[e_{\lambda}^{n+1}(x) + \sum_{i=1}^n \left(\binom{n}{i} \binom{i}{k} + \binom{n}{i-1} \binom{i-1}{k} + \binom{n}{i-1} \binom{i-1}{k-1} \right) e_{\lambda}^{n+1+k-i}(x) e_{\lambda}^i(c) \right. \\
&\quad \left. + \left(\binom{n}{k} + \binom{n}{k-1} \right) e_{\lambda}^k(x) e_{\lambda}^{n+1}(c) \right] + \lambda^{n+1} \sum_{i=0}^{n+1} \binom{n}{i-1} \binom{i-1}{n} e_{\lambda}^{2n+2-i}(x) e_{\lambda}^i(c) \\
&= \sum_{i=0}^{n+1} \binom{n+1}{i} e_{\lambda}^{n+1-i}(x) e_{\lambda}^i(c) + \sum_{k=1}^n \lambda^k \sum_{i=0}^{n+1} \binom{n+1}{i} \binom{i}{k} e_{\lambda}^{n+1+k-i}(x) e_{\lambda}^i(c) \\
&\quad + \lambda^{n+1} \sum_{i=0}^{n+1} \binom{n+1}{i} \binom{i}{n+1} e_{\lambda}^{2n+2-i}(x) e_{\lambda}^i(c) \\
&= \sum_{k=0}^{n+1} \lambda^k \sum_{i=0}^{n+1} \binom{n+1}{i} \binom{i}{k} e_{\lambda}^{n+1+k-i}(x) e_{\lambda}^i(c).
\end{aligned}$$

Thus Eq. (7) is proved. ■

Thus we can use the λ -binomial pseudo-basis $\mathbf{q} = \{e_{\lambda}^n(x)\}$ of $C[[x]]$ and the λ -divided power pseudo-basis $\tau_n(t) = \frac{t(t-a) \cdots (t-a(n-1))}{n!}$, $n \geq 0$, of $C[[t]]$ to define the pairing

$$\langle \cdot | \cdot \rangle_{\lambda}: U_{\lambda}C \times C \langle \mathbf{q} \rangle \rightarrow C$$

as is described in Eq.(2).

Definition 4.4 Let $f(t) \in C[[t]]$ be a delta series. The dual basis of $\{\tau_n(f(t))\}_n$ in $C \langle \mathbf{q} \rangle$ is called the **associated sequence** for $f(t)$.

We show that every λ -binomial sequence is associated to a delta power series, as in the case when $\lambda = 0$.

Theorem 4.5 Let C be a \mathbb{Q} -algebra. Let $\{s_n(x)\}_n$ be a basis of $C \langle \mathbf{q} \rangle$. Then $\{s_n(x)\}$ is a sequence of λ -binomial type if and only if $\{s_n(x)\}$ is the associated sequence of a delta series $f(t)$ in $C[[t]]$.

Proof: If $\{s_n(x)\}$ is the associated sequence of a delta series $f(t)$, then by Theorem 3.10 and Proposition 4.2, $\{s_n(x)\}$ is a λ -binomial sequence.

Conversely, let $\{s_n(x)\}$ be a λ -binomial sequence and let $\{f_n(t)\}$ be the dual basis of $\{s_n(x)\}$ in $C[[t]]$. It follows from Theorem 3.10 that $\{f_n(t)\}$ is a λ -divided power series. We only need to show

1. $f_1(t)$ is a delta series, and
2. $f_n(t) = \tau_n(f_1)(t), \forall n \geq 0$.

We first prove (2) by induction. Since $\{f_n(t)\}$ is a λ -divided power basis, we have

$$f_m(t)f_n(t) = \sum_{k=0}^m \binom{m+n-k}{m} \binom{m}{k} \lambda^k f_{m+n-k}(t).$$

Taking $m = n = 0$, we have $f_0(t)f_0(t) = f_0(t)$. Thus $f_0(t) = 1$. Assuming $f_n(t) = \tau_n(f_1)(t)$, from $f_n(t)f_1(t) = f_{n+1}(t) + n\lambda f_n(t)$ we have

$$f_{n+1}(t) = f_n(t)(f_1(t) - n\lambda) = \tau_n(f_1)(t)(f_1(t) - n\lambda) = \tau_{n+1}(f_1)(t).$$

This proves (2). Then (1) follows since if $f_1(t)$ is not a delta series, then $\{\tau_n(f)(t)\}$ cannot be a pseudo-basis of $C[[t]]$. ■

4.2 Compatibility of λ -calculus

We now show that, for any given $\lambda \in C$, the λ -umbral calculus is compatible with the classical umbral calculus (when $\lambda = 0$). The compatibility is in the following sense. Since $\{x^n\}$ and $\mathbf{q} = \{q_n\}$ are pseudo-basis of $C[[x]]$, the pairings

$$\langle \mid \rangle_0: C[t] \otimes C[x] \rightarrow C$$

and

$$\langle \mid \rangle_\lambda: C[t] \otimes C[\mathbf{q}] \rightarrow C$$

extend to pairings

$$[\mid]_0: C[t] \otimes C[[x]] \rightarrow C$$

and

$$[\mid]_\lambda: C[t] \otimes C[[x]] \rightarrow C.$$

Definition 4.6 We say that $\langle \mid \rangle_0$ and $\langle \mid \rangle_\lambda$ are compatible if

$$[\mid]_0 = [\mid]_\lambda.$$

Theorem 4.7 *The pairings*

$$\langle \mid \rangle_0: C[t] \otimes C[[x]] \rightarrow C$$

and

$$\langle \mid \rangle_\lambda: C[t] \otimes C[[x]] \rightarrow C$$

are compatible.

This theorem has the following immediate consequence.

Corollary 4.8 *Let C be a \mathbb{Q} -algebra. A pseudo-basis $\{p_n(x)\}$ of $C[[x]]$ is of λ -binomial type if and only if it is the dual basis of a λ -divided power pseudo-basis of $C[[t]]$ under the pairing $\langle \mid \rangle_0$.*

Proof of Theorem 4.7: Since $\{\tau_{\lambda,n}(t)\}_n$ is a basis of $C[t]$ and $\{e_\lambda^k(x)\}_k$ is a pseudo-basis of $C[[x]]$, we only need to prove

$$[\tau_{\lambda,n}(t) \mid e_\lambda^k(x)]_0 = \delta_{n,k}, \quad n, k \geq 0. \quad (8)$$

We will prove this equation by induction on n . When $n = 0$, $\tau_{\lambda,n}(t) = 1$. Since $\deg e_\lambda(x) = 1$, we have $[\tau_{\lambda,0}(t) \mid e_\lambda^k(x)]_0 = \delta_{0,k}$. Assume that Eq. (8) holds for an $n \geq 0$ and all $k \geq 0$. Then we have

$$\begin{aligned} & [\tau_{\lambda,n+1}(t) \mid e_\lambda^k(x)]_0 \\ &= \left[\frac{t}{n+1} \tau_{\lambda,n}(t) - \frac{n\lambda}{n+1} \tau_{\lambda,n}(t) \mid e_\lambda^k(x) \right]_0 \\ &= \frac{1}{n+1} [t \tau_{\lambda,n}(t) \mid e_\lambda^k(x)]_0 - \frac{n\lambda}{n+1} [\tau_{\lambda,n}(t) \mid e_\lambda^k(x)]_0 \\ &= \frac{1}{n+1} [t \tau_{\lambda,n}(t) \mid e_\lambda^k(x)]_0 - \frac{n\lambda}{n+1} \delta_{n,k}. \end{aligned}$$

By [Rom, Theorem 2.2.5], we have

$$[t \tau_{\lambda,n}(t) \mid e_\lambda^k(x)]_0 = [\tau_{\lambda,n}(t) \mid \frac{d}{dx} e_\lambda^k(x)]_0.$$

Since

$$\frac{d}{dx} e_\lambda^k(x) = k e_\lambda^{k-1}(x) e^{\lambda x} = \lambda k e_\lambda^k(x) + k e_\lambda^{k-1}(x),$$

we have

$$\begin{aligned} & [\tau_{\lambda,n+1}(t) \mid e_\lambda^k(x)]_0 \\ &= \frac{1}{n+1} [\tau_{\lambda,n}(t) \mid \lambda k e_\lambda^k(x) + k e_\lambda^{k-1}(x)]_0 - \frac{n\lambda}{n+1} \delta_{n,k} \end{aligned}$$

$$\begin{aligned}
&= \frac{\lambda k}{n+1} \delta_{n,k} + \frac{k}{n+1} \delta_{n,k-1} - \frac{n\lambda}{n+1} \delta_{n,k} \\
&= \frac{\lambda(k-n)}{n+1} \delta_{n,k} + \frac{k}{n+1} \delta_{n,k-1} \\
&= \delta_{n+1,k}.
\end{aligned}$$

This completes the induction. ■

Acknowledgement: The author thanks William Keigher for helpful discussions.

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